Math 121 2.6-2nd Chain Rule with Quotient or Product Rule

Objectives

1) Use the chain rule and product rule together in more complicated derivatives.

2) Use the chain rule and quotient rule together in more complicated derivatives.

3) Use multiple chain rules (nested functions)

Find derivatives. Fully factor results if possible.

① $f(x) = x^2\sqrt{x^2-1}$

product of x^2 and $(x^2-1)^{1/2}$

$$f(x) = x^2 \cdot (x^2-1)^{1/2}$$

$$f'(x) = x^2 \cdot \underbrace{\frac{d}{dx}(x^2-1)^{1/2}}_{\text{need chain rule}} + \frac{d}{dx}(x^2)\cdot(x^2-1)^{1/2}$$

$$f'(x) = x^2 \cdot \underbrace{\tfrac{1}{2}(x^2-1)^{-1/2}}_{\text{evaluate derivative of outside at the same inside (compose)}} \cdot \underbrace{2x}_{\text{multiply by derivative of inside.}} + 2x\cdot(x^2-1)^{1/2}$$

Factor GCF = Factor Least Powers

Factor out Lower power $(-\frac{1}{2})$

$$f'(x) = x^3(x^2-1)^{-1/2} + 2x(x^2-1)^{1/2}$$

$$= x(x^2-1)^{-1/2}\left[x^{3-1}(x^2-1)^{-\frac{1}{2}-(-\frac{1}{2})} + 2x^{1-1}(x^2-1)^{\frac{1}{2}-(-\frac{1}{2})}\right]$$

when we factor out, we subtract the exponent we factored out.
This might be subtract a negative.

$$= x(x^2-1)^{-1/2}\left[x^2 + 2\cdot(x^2-1)\right]$$

$$= x(x^2-1)^{-1/2}(x^2+2x^2-2)$$

$$= \boxed{x(x^2-1)^{-1/2}(3x^2-2)} = \boxed{\frac{x(3x^2-2)}{\sqrt{x^2-1}}}$$

② $f(x) = \dfrac{x}{x^2+1}$

find $f''(x)$.

Need quotient rule:

$$f'(x) = \frac{(x^2+1)\cdot\frac{d}{dx}(x) - x\cdot\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)\cdot 1 - x(2x)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$f'(x) = \frac{-x^2+1}{(x^2+1)^2}$$

will need chain rule. to take derivative of denominator.

$$f''(x) = \frac{d}{dx}\left(\frac{-x^2+1}{(x^2+1)^2}\right)$$

Quotient Rule

$$= \frac{(x^2+1)^2\cdot\frac{d}{dx}(-x^2+1) - (-x^2+1)\cdot\frac{d}{dx}\left([x^2+1]^2\right)}{[(x^2+1)^2]^2}$$

chain rule in { }

$$= \frac{(x^2+1)^2\cdot(-2x) - (-x^2+1)\cdot\left\{2[x^2+1]^1\cdot\frac{d}{dx}(x^2+1)\right\}}{(x^2+1)^{2\cdot 2}}$$

derivative of inside

$$= \frac{\overbrace{\boxed{-2x(x^2+1)^2}}^{\text{1st term}} - \overbrace{\boxed{(-x^2+1)\cdot 2(x^2+1)\cdot(2x)}}^{\text{2nd term}}}{(x^2+1)^4}$$

Notice (x^2+1) can be factored out!

$$= \frac{\cancel{(x^2+1)}\left[-2x(x^2+1) - (-x^2+1)\cdot 2\cdot 2x\right]}{\cancel{(x^2+1)}\cdot(x^2+1)^3}$$

and canceled out!

$$= \frac{-2x(x^2+1) - 4x(-x^2+1)}{(x^2+1)^3}$$

$$= \frac{-2x^3-2x+4x^3-4x}{(x^2+1)^3}$$

$$f''(x) = \frac{2x^3 - 6x}{(x^2+1)^3}$$

$$\boxed{f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}}$$

- nothing else cancels
- Leave final answer factored.

③ $f(x) = [(x^3+1)^2 - x]^4$

outermost function u^4

$$f'(x) = \underbrace{4[(x^3+1)^2 - x]^3}_{\frac{d}{du}(u^4) \text{ evaluated at same inside}} \cdot \frac{d}{dx}[\underbrace{(x^3+2)^2}_{\text{another chain rule}} - x]$$

$$= 4[(x^3+1)^2 - x]^3 \cdot [\underbrace{2(x^3+2) \cdot \frac{d}{dx}(x^3+2)}_{\text{chain rule}} - \underbrace{1}_{\text{power rule}}]$$

$$= 4[(x^3+1)^2 - x]^3 \cdot [2(x^3+2) \cdot (3x^2+0) - 1]$$

$$= 4[\underbrace{(x^3+1)^2}_{\text{FOIL}} - x]^3 \cdot [\underbrace{2 \cdot 3x^2(x^3+2)}_{\text{dist}} - 1] = 4[(x^3+1)^2 - x]^3 \cdot [6x^2(x^3+2) - 1]$$

$$= 4[x^6 + 2x^3 + 1 - x]^3 [6x^5 + 12x^2 - 1]$$

$$= 4(x^6 + 2x^3 - x + 1)^3 (6x^5 + 12x^2 - 1)$$

④ Find the second derivative of $f(x) = (x^3-1)^5$

$$f'(x) = \underbrace{5(x^3-1)^4}_{\text{deriv. outside eval. @ inside}} \cdot \frac{d}{dx}(x^3-1)$$

$$= 5(x^3-1)^4 \cdot (3x^2)$$

$$f'(x) = 15x^2 (x^3-1)^4$$

To find f'' we will need the Product Rule

$$f'(x) = \underbrace{15x^2}_{\text{contains } x} \cdot \underbrace{(x^3-1)^4}_{\text{contains } x}$$

$$f''(x) = \frac{d}{dx}(15x^2)\cdot(x^3-1)^3 + \underbrace{\frac{d}{dx}(x^3-1)^3}_{\text{will need Chain Rule}}\cdot 15x^2$$

$$= 30x^2(x^3-1)^3 + \underbrace{3(x^3-1)^2}_{\substack{\frac{d}{dx}\text{ outside}\\ \text{eval}\\ \text{@ inside}}}\cdot\underbrace{\frac{d}{dx}(x^3-1)}_{\frac{d}{dx}\text{ inside}}\cdot 15x^2$$

$$= 30x^2(x^3-1)^3 + 3(x^3-1)^2\cdot(3x^2+0)\cdot 15x^2$$

$$= \underbrace{\boxed{30x^2(x^3-1)^3}}_{\text{1st term}} + \underbrace{\boxed{3\cdot 3x^2\cdot 15x^2(x^3-1)^2}}_{\text{2nd term}}$$

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Factor GCF/ Least Powers $5x^2(x^3-1)^2$

$$= 15x^2(x^3-1)^2\left[\underbrace{2(x^3-1)}_{\text{1st}} + \underbrace{9x^2}_{\text{2nd}}\right]$$

$$= 15x^2(x^3-1)^2\left[2x^3-2+9x^2\right]$$

$$= \boxed{15x^2(x^3-1)^2(2x^3+9x^2-2)}$$

(5) Find the derivative. $f(x) = \dfrac{2x^3-7x}{\sqrt[3]{(2x^2-3x+5)^2}}$

Rewrite radical $\sqrt[3]{u^2} = u^{2/3}$

$$f(x) = \frac{2x^3-7x}{(2x^2-3x+5)^{2/3}} = (2x^3-7x)(2x^2-3x+5)^{-2/3}$$

Quotient Rule Needed Option 2

Product Rule Needed Option 1

Negative exponent

Option 1: $\dfrac{d}{dx}\left[(2x^3-7x)(2x^2-3x+5)^{-2/3}\right]$

Product Rule

$$f'(x) = \frac{d}{dx}(2x^3-7x)\cdot(2x^2-3x+5)^{-2/3} + \frac{d}{dx}(2x^2-3x+5)^{-2/3}\cdot(2x^3-7x)$$

$$= \underbrace{(6x^2-7)\boxed{(2x^2-3x+5)^{-2/3}}}_{\text{1st term}} + \underbrace{\underbrace{\left(-\tfrac{2}{3}\right)\boxed{(2x^2-3x+5)^{-5/3}}(4x-3)}_{\text{chain rule}}\cdot(2x^3-7x)}_{\text{2nd term}}$$

Factor Least Powers $(2x^2-3x+5)^{-5/3}$ b/c $-5/3 < -2/3$

$$f'(x) = (2x^2-3x+5)^{-5/3}\left[(6x^2-7)(2x^2-3x+5)^{-2/3-(-5/3)} - \frac{2}{3}(4x-3)(2x^3-7x)\right]$$

$$= (2x^2-3x+5)^{-5/3}\left[(6x^2-7)(2x^2-3x+5) - \frac{2}{3}(4x-3)(2x^3-7x)\right]$$

$$-\frac{2}{3}+\frac{5}{3} = \frac{3}{3} = 1$$

mult

Foil

$$= (2x^2-3x+5)^{-5/3}\left[12x^4-18x^3+30x^2-14x^2+21x-35 \quad -\frac{2}{3}(8x^4-28x^2-6x^3+21x)\right]$$

$$= (2x^2-3x+5)^{-5/3}\left[12x^4-18x^3+16x^2+21x-35-\frac{16}{3}x^4+4x^3+\frac{56}{3}x^2-14x\right]$$

combine

dist

$$\boxed{= (2x^2-3x+5)^{-5/3}\left[\frac{20}{3}x^4-14x^3+\frac{104}{3}x^2+7x-35\right]}$$

$$= \frac{\left(\frac{20}{3}x^4-14x^3+\frac{104}{3}x^2+7x-35\right)\cdot 3}{\left[(2x^2-3x+5)^{5/3}\right]\cdot 3}$$

clear frac

$$= \boxed{\frac{20x^4-42x^3+104x^2+21x-105}{3(2x^2-3x+5)^{5/3}}}$$

$$= \boxed{\frac{20x^4-42x^3+104x^2+21x-105}{3\sqrt[3]{(2x^2-3x+5)^5}}}$$

$$= \boxed{\frac{1}{3}(2x^2-3x+5)^{-5/3}(20x^4-42x^3+104x^2+21x-105)}$$

Option 2: Quotient Rule

chain rule

$$f'(x) = \frac{(2x^2-3x+5)^{2/3}\cdot\frac{d}{dx}(2x^3-7x) - (2x^3-7x)\cdot\frac{d}{dx}(2x^2-3x+5)^{2/3}}{((2x^2-3x+5)^{2/3})^2}$$

$$= \frac{\boxed{(2x^2-3x+5)^{2/3}}(6x^2-7) - (2x^3-7x)\cdot\frac{2}{3}\boxed{(2x^2-3x+5)^{-1/3}}(4x-3)}{(2x^2-3x+5)^{4/3}}$$

mult exp

$=$ Factor least powers $(2x^2-3x+5)^{-1/3}$ because $-1/3 < 2/3$.

$$= \frac{(2x^2-3x+5)^{-1/3}\left[(2x^2-3x+5)^{2/3-(-1/3)}(6x^2-7) - \frac{2}{3}(2x^3-7x)(4x-3)\right]}{(2x^2-3x+5)^{4/3}}$$

exp $2/3+1/3=1$

$$= \frac{\left[(2x^2-3x+5)(6x^2-7) - \frac{2}{3}(2x^3-7x)(4x-3)\right]}{(2x^2-3x+5)^{1/3}(2x^2-3x+5)^{4/3}}$$

pos exp in denom

$$= \frac{\left[12x^4-14x^2-18x^3+21x+30x^2-35-\frac{2}{3}(8x^4-6x^3-28x^2+21x)\right]}{(2x^2-3x+5)^{4/3+1/3}}$$

$$= \frac{12x^4-18x^3+16x^2+21x-35-\frac{16}{3}x^4+4x^3+\frac{56}{3}x^2-14x}{(2x^2-3x+5)^{5/3}}$$

$$= \frac{\frac{20}{3}x^4-14x^3+\frac{104}{3}x^2+7x-35}{(2x^2-3x+5)^{5/3}}$$

simplify as before in Option 1.